

# A Topological Field Theory for the triple Milnor linking coefficient

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The subject of this work is a three-dimensional topological field theory with a non-semisimple group of gauge symmetry and observables consisting in the holonomies of connections around three closed loops. The connections are a linear combination of gauge potentials with coefficients containing a set of one-dimensional scalar fields. It is checked that these observables are both metric independent and gauge invariant. The gauge invariance is achieved by requiring non-trivial gauge transformations in the scalar field sector. This topological field theory is solvable and has only a relevant amplitude which has been computed exactly. From this amplitude it is possible to isolate a topological invariant which is Milnor's triple linking coefficient. The topological invariant obtained in this way is in the form of a sum of multiple contour integrals. The contours coincide with the trajectories of the three loops mentioned before. The introduction of the one-dimensional scalar field is necessary in order to reproduce correctly the particular path ordering of the integration over the contours which is present in the triple Milnor linking coefficient. This is the first example of a local topological gauge field theory that is solvable and can be associated to a topological invariant of the complexity of the triple Milnor linking coefficient. After eliminating the scalar fields, the topological field theory of [1] is recovered. This model is consistent whenever any pair of loops has vanishing Gauss linking number and the gauge potentials are invariant under abelian gauge transformations. Some of its observables are not local. With the addition of the one-dimensional scalar fields a new topological field theory has been achieved that is consistent independently of the way in which the loops are entangled and the observables are local. Moreover, the

gauge symmetry is much richer, because it is based on a non-semisimple non-abelian group.

## I. INTRODUCTION

The correspondence between three-dimensional topological field theories and topological invariants of knots and links is well known after the seminal paper of [2]. From the amplitudes of Chern–Simons field theories it is for instance possible to isolate invariants which are in the form of a sum of multiple contour integrals [3, 4]. Each contour integral appearing in invariants of this kind may be explicitly represented as follows:

$$\mathcal{I} = \int_{a_1}^{b_1} ds_1 \cdots \int_{a_n}^{b_n} ds_n f(s_1, \dots, s_n) \quad (1)$$

where  $s_1, \dots, s_n$  represent the variables that parametrize the closed contours. Often the integration is path ordered, which means that for some of the pairs  $s_i, s_j$  of variables it is requested that  $s_i \leq s_j$ . In this work the attention will be focused on invariants of this type, which are particular cases of the so-called numerical knot and link invariants. The problem that will be addressed here and that has been formulated in much more details in Ref. [5], can be summarized by the following fundamental question:

*Given a particular numerical knot or link invariant expressed as a sum of multiple contour integrals, is it possible to find a solvable and local topological field theory, characterized by an amplitude that is proportional to that invariant or to a function of it?*

That posed above is not just a theoretical question. Topological field theories that can be associated only to a particular invariant are one of the main tools in studying the physics of knotted and linked polymer rings [6]. Applications can also be found in other systems in which quasi one-dimensional ring-shaped objects play a relevant role. This is the case of magnetic lines on the surface of the Sun, which are heavily entangled and give rise to complicated topological configurations [7]. As the observations point out, the probability of a coronal mass ejection is growing with the increasing of the topological complexity [8, 9].

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An invariant associated to a topological field theory with a particular non-semisimple group of local symmetries derived in [10] was independently studied in connection with the solar magnetic fields in [11]. Many other invariants can be obtained simply by choosing different non-semisimple groups. An example of this strategy can be found in Ref. [5], in which an invariant describing the topological states of a link composed by four knots has been derived.

While topological field theories with non-semisimple gauge groups have been very successful in establishing a correspondence between numerical link invariants and topological field theories, an important issue is still left unsolved. Up to now, in fact, it was possible to obtain only link invariants containing contour integrals that are not path-ordered. Unfortunately, all the known knot invariants and most of the link invariants that can be cast in the form of multiple contour integrals require path-ordering.

A breakthrough toward the solution of this issue is the work of Ref. [1], in which the case of the triple Milnor linking coefficient  $\bar{\mu}(1, 2, 3)$  has been treated. The only drawback of the topological field theory constructed in [1] is that some of its observables are non-local, because they contain a bilocal vector density. For this reason, the theory cannot be easily applied to polymer physics and its full gauge symmetry remains hidden. A full gauge symmetry is however required in order to deal with the spurious degrees of freedom due to gauge invariance. To make the model of [1] local, we start from a simple observation. Let us consider the path-ordered double integral  $A = \int_a^b ds \int_a^s dt f(s, t)$ . With the help of a Heaviside  $\theta$ -function,  $A$  can be rewritten in the form  $A = \int_a^b ds \int_a^b dt \theta(s - t) f(s, t)$ . The crucial point is that the  $\theta$ -function is the propagator of the "topological" one-dimensional field theory  $S_\alpha = \int_{-\infty}^{+\infty} d\eta \alpha(\eta) \frac{d\alpha(\eta)}{d\eta}$ . This theory is topological in the sense that it is invariant under reparametrization of the infinite line  $\mathbb{R}$ . We show here that the topological field theory with non-local observables of [1] may be converted into a local one thanks to the introduction of a suitable set of  $\alpha$ -fields. It has been possible to prove that this local version is invariant under a non-semisimple gauge group of symmetry like the theories discussed in [5, 10]. Using the results of these previous works, in particular the fact that for gauge transformations like those considered here the Faddeev-Popov determinant is trivial, the partition function of the topological field theory associated to the triple Milnor linking coefficient is explicitly computed.

## II. CONVENTIONS

In the following Greek letters  $\mu, \nu, \rho, \dots = 1, 2, 3$  will be used to denote the spatial indices on the flat three dimensional space  $\mathbb{R}^3$ . The position of a point  $x$  on  $\mathbb{R}^3$  will be given by specifying its cartesian coordinates  $x^\mu$ . Latin letters  $i, j, k, \dots = 1, 2, 3$  will be reserved for the indices of the internal symmetries and for labeling the three closed trajectories  $P_1, P_2, P_3$ . Throughout this paper the convention of summing over repeated indices will be followed. In the case in which this will not be possible, barred indices  $\bar{i}, \bar{j}, \bar{k}, \dots = 1, 2, 3$  will be adopted. For instance,  $A^{\bar{i}}B_{\bar{i}}$  is the product of the  $\bar{i}$ -th components of the vectors  $\mathbf{A}$  and  $\mathbf{B}$ , while their vector product is defined as:  $\mathbf{A} \cdot \mathbf{B} = A^i B_i \equiv \sum_{\bar{i}=1}^3 A^{\bar{i}} B_{\bar{i}}$  where the symbol  $\equiv$  denotes equivalence.

The trajectories  $P_i, i = 1, 2, 3$ , will be represented by curves  $x_i^\mu(s)$  parametrized by means of their arc-lengths. It will be assumed that all three loops have the same length  $L$ , so that  $0 \leq s \leq L$ .

For the formulation of the topological field theory presented here the two triplets of vector fields  $A_\mu^i(x)$  and  $a_{i\mu}$  will be needed. In addition, we introduce the set of one-dimensional scalar field  $\alpha_{ij}(\eta)$ , where  $-\infty < \eta < +\infty$ . For future purposes we define also the following three external sources:

$$T_i^{\mu x} = \oint_{P_i} dx_i^\mu \delta^{(3)}(x - x_i) = \int_0^L ds \dot{x}_i^\mu(s) \delta^{(3)}(x - x_i(s)) \quad (2)$$

$$T_i^{\{\mu x, \nu y\}} = \oint_{P_i} dx_i^\mu \int_0^x dx_i'^\nu \delta^{(3)}(x - x_i) \delta^{(3)}(y - x'_i) \quad (3)$$

and

$$\xi_{ij}(\eta) = \int_0^L ds \delta(\eta - s) \dot{x}_i^\mu(s) a_{j\mu}(x_i(s)) \quad (4)$$

Finally, the expression of the triple Milnor linking coefficient  $\bar{\mu}(1, 2, 3)$  [12], which is able to distinguish the topological states of a link composed by three loops, is provided below [13]:

$$\bar{\mu}(1, 2, 3) = -\frac{1}{2} \int d^3x \epsilon^{\mu\nu\rho} \tilde{a}_{1\mu} \tilde{a}_{2\nu} \tilde{a}_{3\rho} + \frac{1}{2} \epsilon^{ijk} \int d^3x \int d^3y T_i^{\{\mu x, \nu y\}} \tilde{a}_{j\mu}(x) \tilde{a}_{k\nu}(y) \quad (5)$$

where  $\epsilon^{\mu\nu\rho}$  and  $\epsilon^{ijk}$  are completely antisymmetric tensors satisfying the convention  $\epsilon^{123} = 1$  and

$$\tilde{a}_{i\mu}(x) = \frac{\epsilon_{\mu\nu\rho}}{4\pi} \int_0^L ds \dot{y}_i^\rho(s) \frac{(x - y_i(s))^\nu}{|x - y_i(s)|^3} \quad (6)$$

Using the definition of the bilocal density  $T_i^{\{\mu x, \nu y\}}$  of Eq. (3), the Milnor linking coefficient may be explicitly expressed in terms of contour integrals over the loops  $P_i$ :

$$\bar{\mu}(1, 2, 3) = -\frac{1}{2} \int d^3x \epsilon^{\mu\nu\rho} \tilde{a}_{1\mu} \tilde{a}_{2\nu} \tilde{a}_{3\rho} + \frac{1}{2} \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_0^L ds \dot{x}_{\bar{i}}^\mu(s) \int_0^s dt \dot{x}_{\bar{i}}^\nu(t) \tilde{a}_{j\mu}(x_{\bar{i}}(s)) \tilde{a}_{k\nu}(x_{\bar{i}}(t)) \quad (7)$$

It is easy to check that the quantity  $\bar{\mu}(1, 2, 3)$  in Eq. (7) coincides up to an overall constant factor with the quantity  $S^1(1, 2, 3)$  appearing in Eq. (16) of Ref. [1].

### III. THE TOPOLOGICAL FIELD THEORY

Let us consider the topological field theory defined by the action

$$S = \int d^3x \epsilon^{\mu\nu\rho} \left\{ 4A_\mu^i \partial_\nu a_{i\rho} + \frac{2}{3} \lambda \epsilon^{ijk} a_{i\mu} a_{j\nu} a_{k\rho} \right\} - 2 \int d^3x T_i^{\mu x} A_\mu^i(x) + 2\lambda \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_{-\infty}^{+\infty} d\eta \left[ \frac{\alpha_{\bar{i}j} \dot{\alpha}_{\bar{i}k}}{2} - \xi_{\bar{i}j} \alpha_{\bar{i}k} \right] \quad (8)$$

The above action is manifestly invariant under diffeomorphisms in the  $\mathbb{R}^3$  space and under reparametrizations of the  $\eta$  variable. In addition, it is possible to show that  $S$  is invariant under the set of gauge transformations:

$$A_\mu^i(x) \longrightarrow A_\mu^i(x) + \partial_\mu \Omega^i(x) + \lambda \left( \frac{1}{2} \omega_j(x) \partial_\mu \omega_k(x) + \omega_j(x) a_{k\mu}(x) \right) \epsilon^{ijk} \quad (9)$$

$$a_{i\mu}(x) \longrightarrow a_{i\mu}(x) + \partial_\mu \omega_i(x) \quad (10)$$

$$\alpha_{ij}(\eta) \longrightarrow \alpha_{ij}(\eta) - \int_0^L ds \theta(\eta - s) \frac{d\omega_j(x_i(s))}{ds} \quad (11)$$

Here  $\theta(\eta - s)$  is the Heaviside theta function and the  $\omega_i(x)$ 's are arbitrary functions of the point  $x \in \mathbb{R}^3$ . The functions  $\Omega^i(x)$  may be split into a single-valued and a multi-valued contribution as follows:

$$\Omega^i(x) = \Omega_s^i(x) + \Omega_{m1}^i(x) + \Omega_{m2}^i(x) \quad (12)$$

The  $\Omega_s^i(x)$ 's,  $i = 1, 2, 3$  are single-valued functions, while the  $\Omega_{m1}^i(x)$ 's and the  $\Omega_{m2}^i(x)$ 's have a special form dictated by the requirement of gauge invariance:

$$\Omega_{m1}^{\bar{i}}(x) = 2\lambda \epsilon^{\bar{i}jk} \int_{x_{\bar{i}}(0)}^x dz_{\bar{i}}^\mu(t) \left[ a_{j\mu}(z_{\bar{i}}(t)) + \frac{1}{2} \frac{\partial \omega_j(z_{\bar{i}}(t))}{\partial z_{\bar{i}}^\mu(t)} \right] [\omega_k(z_{\bar{i}}(t)) - \omega_k(z_{\bar{i}}(0))] \quad (13)$$

$$\Omega_{m2}^{\bar{i}}(x) = -\lambda \epsilon^{\bar{i}jk} \int_{x_{\bar{i}}(0)}^x dz_{\bar{i}}^\mu(t) \left[ a_{\mu k}(z_{\bar{i}}(t)) + \frac{1}{2} \frac{\partial \omega_k(z_{\bar{i}}(t))}{\partial z_{\bar{i}}^\mu(t)} \right] \omega_j(z_{\bar{i}}(t)) \quad (14)$$

In Eqs. (13) and (14)  $z_i^\mu(t)$  is an arbitrary curve joining the point  $x_i(0)$  belonging to the loop  $P_i$  to the generic point  $x \in \mathbb{R}^3$ . It is easy to check that  $\Omega_{m1}^i(x)$  and  $\Omega_{m2}^i(x)$  are multi-valued functions depending on the choice of the trajectory  $z_i^\mu(t)$ . We note also that  $\Omega_{m2}^i(x)$  satisfies the identity:

$$\oint_{P_i} dx_i^\mu \left[ \partial_\mu \Omega_{m2}^{\bar{i}} + \lambda \epsilon^{\bar{i}jk} \left( \frac{1}{2} \omega_j \partial_\mu \omega_k + \omega_j a_{\mu k} \right) \right] = 0 \quad (15)$$

In words, the above relation means that the non-linear contributions in the  $\omega_i$ 's appearing in the transformation (9) of the fields  $A_\mu^i(x)$  is exactly canceled by the contribution due to  $\Omega_{m2}^i(x)$  when the fields  $A_\mu^i$  are integrated along the loops  $P_i$ . To prove the invariance of the action (8) under the gauge transformations (9)–(11) it is convenient to split  $S$  into four contributions:

$$S = S_{top} + S_\alpha + S_{source1} + S_{source2} \quad (16)$$

where  $S_{top}$  is the "bulk action" on  $\mathbb{R}^3$

$$S_{top} = \int d^3x \epsilon^{\mu\nu\rho} \left\{ 4A_\mu^i \partial_\nu a_{i\rho} + \frac{2}{3} \lambda \epsilon^{ijk} a_{i\mu} a_{j\nu} a_{k\rho} \right\} \quad (17)$$

and  $S_\alpha$  is the action of the one-dimensional fields  $\alpha_{ij}(\eta)$ :

$$S_\alpha = \lambda \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_{-\infty}^{+\infty} d\eta \alpha_{\bar{i}j} \dot{\alpha}_{\bar{i}k} \quad (18)$$

Finally,  $S_{source1}$  and  $S_{source2}$  take into account the source terms in Eq. (8) and may be explicitly written as follows:

$$S_{source1} = -2 \sum_{\bar{i}=1}^3 \int_0^L ds \dot{x}_i^\mu(s) A_\mu^{\bar{i}}(x_{\bar{i}}(s)) \quad (19)$$

$$S_{source2} = -2\lambda \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_0^L ds \dot{x}_i^\mu(s) a_{j\mu}(x_{\bar{i}}(s)) \alpha_{\bar{i}k}(s) \quad (20)$$

It is possible to check that  $S_{top}$  is fully gauge invariant, i. e.:

$$S_{top} \longrightarrow S_{top} \quad (21)$$

To prove Eq. (21) we have used the fact that the gauge fields  $a_{i\mu}$  and  $A_\mu^i$  vanish at infinity sufficiently fast and the identity  $\epsilon^{\mu\nu\rho} \int d^3x \Omega^i \partial_\mu \partial_\nu a_{i\rho} = 0$ . This identity is verified because the spatial components of  $a_{i\mu}$  are not multi-valued. Next, we consider the combination

$S_\alpha + S_{source2}$  which is not fully invariant and transforms as shown below:

$$S_\alpha + S_{source2} \longrightarrow S_\alpha + S_{source2} + 2\lambda \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_0^L ds \left[ \dot{x}_{\bar{i}}^\mu(s) a_{j\mu}(x_{\bar{i}}(s)) + \frac{1}{2} \frac{d\omega_j(x_{\bar{i}}(s))}{ds} \right] [\omega_k(x_{\bar{i}}(s)) - \omega_k(x_{\bar{i}}(0))] \quad (22)$$

The unwanted terms violating gauge invariance are canceled exactly by the transformation of the term  $S_{source1}$  if the functions  $\Omega^i(x)$  appearing in Eq. (9) are chosen as in Eqs. (12)-(14). In this case, in fact, a straightforward calculation shows that:

$$S_{source1} \longrightarrow S_{source1} - 2\lambda \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_0^L ds \left[ \dot{x}_{\bar{i}}^\mu(s) a_{j\mu}(x_{\bar{i}}(s)) + \frac{1}{2} \frac{d\omega_j(x_{\bar{i}}(s))}{ds} \right] [\omega_k(x_{\bar{i}}(s)) - \omega_k(x_{\bar{i}}(0))] \quad (23)$$

Eq. (23) has been obtained by taking into account Eq. (15). Clearly, in the action  $S$  of Eq. (16) the non-invariant contributions appearing after a gauge transformation in the right hand side of equations (22) and (23) vanish identically. As a conclusion, the action  $S$  is gauge invariant.

#### IV. THE MILNOR INVARIANT

In the rest of this Section we will concentrate our attention to the partition function of the theory, which is given by:

$$\mathcal{Z} = \int \mathcal{D}\alpha_{ij} \mathcal{D}a_{i\mu} \mathcal{D}A_\mu^i e^{-iS} \quad (24)$$

The above gauge field theory requires the introduction of a gauge fixing, like for instance the Lorentz gauge. It has been shown in [10] that nonlinear gauge transformations like those of Eq. (9) give rise to trivial Faddeev-Popov determinants, in which the ghosts are decoupled from the gauge fields. For this reason, the ghost sector can be ignored.

The connection with Ref. [1] is obtained after eliminating the one-dimensional fields  $\alpha_{ik}$ . After performing a Gaussian integration, it is possible to prove the following identity:

$$\int \mathcal{D}\alpha_{ij} \exp \left\{ -2\lambda i \epsilon^{\bar{i}jk} \sum_{\bar{i}=1}^3 \int_{-\infty}^{+\infty} d\eta \left[ \frac{\alpha_{\bar{i}j} \dot{\alpha}_{\bar{i}k}}{2} - \xi_{\bar{i}j} \alpha_{\bar{i}k} \right] \right\} = e^{2i\lambda I} \quad (25)$$

where

$$I = \sum_{\bar{i}=1}^3 \epsilon^{\bar{i}jk} \int_0^L ds \int_0^L dt \theta(s-t) \dot{x}_{\bar{i}}^\mu(s) \dot{x}_{\bar{i}}^\nu(t) a_{j\mu}(x_{\bar{i}}(s)) a_{k\nu}(x_{\bar{i}}(t)) \quad (26)$$

and the Heaviside function  $\theta(s-t)$  is nothing but the propagator of the fields  $\alpha_{ij}$ . The quantity  $I$  can be cast in the form of a double volume integral in  $\mathbb{R}^3$ :

$$I = \epsilon^{ijk} \int d^3x \int d^3y \epsilon^{ijk} T_i^{\{\mu x, \nu y\}} a_{j\mu}(x) a_{k\nu}(y) \quad (27)$$

Thus, the partition function in Eq. (24) may be rewritten as follows:

$$\mathcal{Z} = \int \mathcal{D}A_\mu^i \mathcal{D}a_{i\mu} e^{-iS'} \quad (28)$$

where

$$\begin{aligned} S' = & \int d^3x \epsilon^{\mu\nu\rho} \left\{ 4A_\mu^i \partial_\nu a_{i\rho} + \frac{2}{3} \lambda \epsilon^{ijk} a_{i\mu} a_{j\nu} a_{k\rho} \right\} \\ & - 2 \int d^3x T_i^{\mu x} A_\mu^i(x) + 2\lambda \int d^3x \int d^3y T_i^{\{\mu x, \nu y\}} a_{j\mu}(x) a_{k\nu}(y) \end{aligned} \quad (29)$$

In this way the topological field theory discussed in [1] has been recovered. In the partition function (28) the gauge fields  $A_\mu^i$  are playing the role of Lagrange multipliers imposing the condition:

$$\epsilon^{\mu\nu\rho} \partial_\nu a_{i\rho}(x) = \frac{1}{2} T_i^{\mu x} \quad (30)$$

These fields can be integrated out giving as a result:

$$\mathcal{Z} = \int \mathcal{D}a_{i\mu}(x) e^{-iS''} \delta(4\epsilon^{\mu\nu\rho} \partial_\nu a_{i\rho}(x) - 2T_i^{\mu x}) \quad (31)$$

where

$$S'' = \int d^3x \frac{2}{3} \lambda \epsilon^{\mu\nu\rho} \epsilon^{ijk} a_{i\mu} a_{j\nu} a_{k\rho} + 2\lambda \int d^3x \int d^3y \epsilon^{ijk} T_i^{\{\mu x, \nu y\}} a_{j\mu}(x) a_{k\nu}(y) \quad (32)$$

After a further integration over the fields  $a_{i\mu}$ , the partition function becomes  $\mathcal{Z} = e^{iS''}$ , where  $S''$  is computed at the solutions of Eq. (30). These solutions are nothing but the  $\tilde{a}'_{i\mu}$ s provided in Eq. (6) apart from a proportionality factor  $-2$ . Comparing the form of  $S''$  with that of the Milnor linking coefficient  $\bar{\mu}(1, 2, 3)$  of Eq. (5), it is clear that they coincide and it is possible to conclude that:

$$\mathcal{Z} = e^{2i\lambda\bar{\mu}(1,2,3)} \quad (33)$$



## V. CONCLUSIONS

The topological field theory defined by the action (8) is invariant under diffeomorphisms on  $\mathbb{R}^3$  and reparametrizations of the variable  $\eta \in \mathbb{R}$ . As it has been shown in Section III, it is also gauge invariant. Finally, it is exactly solvable. Its partition function consists essentially in the Milnor linking coefficient  $\bar{\mu}(1, 2, 3)$ . This is the first time that a local topological field theory has been constructed whose partition function can be computed in closed form and is associated to a topological invariant of the complexity of the Milnor linking coefficient. Applications of this topological field theory to polymer physics are currently work in progress.

## VI. ACKNOWLEDGMENTS

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